SMALLEST FLOW RATE WITH A SPECIFIED AREA OF THE FILTRATION REGION

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The forms of regions with a fixed area realizing the minimum filtrational flow rate or total heat flux are found.

Plane steady filtration of a homogeneous incompressible liquid in a homogeneous undeformable porous medium according to a linear or power law is considered. The filtration region is an enlarged stream tube with input and output of the filtrational flux at boundaries. The area of the region and the position of the inlet are specified and the position of the output section (curve Γ) is unknown and determined from the condition of minimum filtrational flow rate. Thereby the solution of the problem gives the guaranteed flow rate with specified area. At the same time, the analogy between filtration and heat conduction means that, in a number of cases, this problem may be regarded as the problem of finding an optimal (in a certain sense) form of the cross section of the heat-source insulation (for example, ensuring a minimum of heat losses with a specified weight of insulation).

In [1, 2], the problem described in terms of linear-filtration theory was reduced to a boundary problem with a free boundary Γ at which the modulus of the velocity v is constant. An analytical solution was obtained for some model examples.

It is shown below that the constraints on the class of solvable problems may be relaxed. This allows the investigation of applied problems of filtration and heat-conduction theories to be expanded.

1. Filtration from an Elliptical Supply Contour

Consider linear filtration from the supply contour AD of elliptical form to a line of constant pressure difference Γ (see Fig. 1a, where a quarter of the flow region G with boundary ABCD is shown). Suppose that the pressure difference h = 0 at AD, h = −H at Γ, and the area of the region is S. It is required to find the form of curve Γ such that a minimum of the filtrational flow rate is realized. According to the foregoing, the following boundary condition is obtained

$$\Delta h = 0,\ x,\ y \in G;\quad h|_{AD} = 0;\quad h|_{\Gamma} = -H,\quad \left.\frac{\partial h}{\partial n}\right|_{\Gamma} = v_0;$$
$$\left.\frac{\partial h}{\partial n}\right|_{CD} = \left.\frac{\partial h}{\partial n}\right|_{AB} = 0. \tag{1}$$

Introducing the region D_u of the auxiliary complex variable $u = \xi + i\eta$, a rectangle with the vertices A(0, 0), B(1, 0), C(1, iα), D(0, iα), letting $z = x + iy$, $\phi = -h$, $W(u) = \phi + i\psi$, and using the analytical function $\Omega(u) = \ln(v_0 dz/dW) = \ln(v_0/v) + i\theta$, it is possible to write dz in the form (see [3], for example)

$$dz = \frac{1}{v_0}\left(\frac{dW}{du}\right)\exp\left[\Omega(u)\right] du. \tag{2}$$

Comparison of the regions of variation of u and the complex potential W(u) leads to the conclusion that dW/du = H. To find the function $\Omega(u) = \nu + i\varepsilon$, the corresponding boundary conditions must first be established in D_u. It is evident from a comparison of Fig. 1 and the form of the region D_u and also from the condition that $v = v_0$ on Γ that

$$\varepsilon|_{AB} = 0,\quad \varepsilon|_{DC} = \pi/2,\quad \nu|_{BC} = 0. \tag{3}$$

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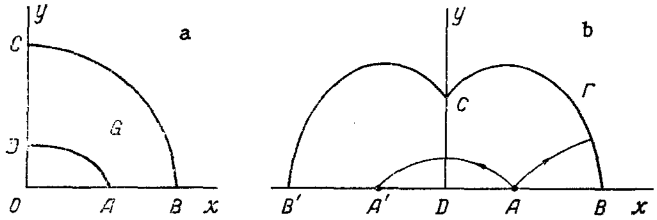

Fig. 1. Diagram of the flow from an elliptical supply contour (a) and from the source to the sink (b).

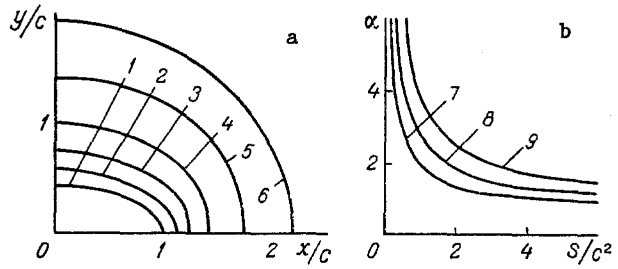

Fig. 2. Form of the filtration region (a) and dependence of the minimum flow rate on its area (b): a) boundary of ellipse with $\mu = 0.9$ (1), $\alpha = 1.5, 2, 3, 5, 10$ (2-6); b) with $\mu = 1, 0.9, 0$ (7-9).

To obtain the boundary condition at AD, it is taken into account that the curvature of the ellipse is specified

$$K(\theta) = \frac{1}{R}(1 - \mu^2 \sin^2\theta)^{3/2}.$$

Using the procedure well known in jet theory [3], the form of the required boundary condition on AD is found

$$\frac{d\varepsilon}{d\eta} = M\,[1 - \mu^2 \sin^2 \varepsilon(\eta)]^{3/2} \exp[\nu(\eta)], \tag{4}$$

where $M = H/(v_0 R)$. Now the boundary problem for the function $\Omega(u)$ with the conditions in Eqs. (3) and (4) must be solved. Its method of solution was described in [3] (see also [4]). Omitting the details, the result is

$$\Omega(u) = \frac{\pi(u-1)}{2\alpha} + \sum_{k=1}^{\infty} c_k \,\mathrm{sh}\left[\frac{\pi k(u-1)}{\alpha}\right], \tag{5}$$

where the coefficients c_k are determined by the iterative method from the formula

$$c_k = \frac{2M}{\pi k \,\mathrm{ch}\,\dfrac{\pi k}{\alpha}} \int_0^{\alpha} \Phi(\eta) \cos\frac{\pi k \eta}{\alpha}\, d\eta, \quad \frac{1}{M} = \frac{2}{\pi}\int_0^{\alpha} \Phi(\eta)\, d\eta,$$

$$\Phi(\eta) = \frac{\left\{1 - \mu^2 \sin^2\left[\dfrac{\pi}{2\alpha}\eta + \displaystyle\sum_{k=1}^{\infty} c_k \,\mathrm{ch}\frac{\pi k}{\alpha} \sin\left(\frac{\pi k}{\alpha}\eta\right)\right]\right\}^{3/2}}{\exp\left[\dfrac{\pi}{2\alpha} + \displaystyle\sum_{k=1}^{\infty} c_k \,\mathrm{sh}\frac{\pi k}{\alpha} \cos\left(\frac{\pi k}{\alpha}\eta\right)\right]}.$$

Substituting Eq. (5) into Eq. (2) and integrating under the condition that $z(0) = 0$, $z = z(u)$ is found and, in particular, the equation of curve Γ. The solution of the problem will depend on the parameter α, which is related to the specified area S of the flow region. This relation may be found by taking account of Eq. (2) and the well-known relation

$$S = \iint_{D_u} \left|\frac{dz}{du}\right|^2 d\xi d\eta.$$

TABLE 1. Dimensionless Values of the Flow Rate and the Area of the Region

Curve number	s	$q/v_0 l$	S/l^2	Curve number	s	$q/v_0 l$	S/l^2
1	0,5	5,69	3,01	4	0,5	11,9	5,44
2	1	7,29	4,49	5	1	17,5	11,4
3	2	9,28	6,96	6	2	24,3	23,7

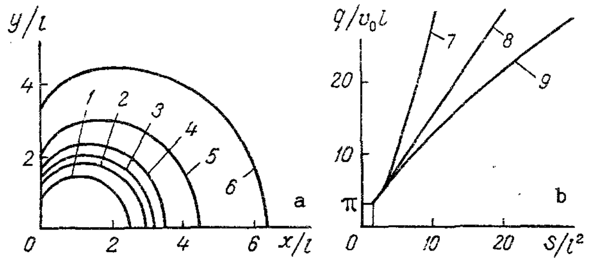

Fig. 3. Form of the filtration region (a) and dependence of the minimum flow rate on its area (b). The notation in (a) is as in Fig. 2; for (b): s = 0.5 (7), 1 (8), and 2 (9).

Numerical results by these formulas lead to the results in Fig. 2. It is evident that at small α, i.e., small flow rates at fixed pressure difference, the desired boundary is close to the supply contour, and it tends to a circle with its center at the coordinate origin with increase in α. When $\mu = 0$, the results obtained lead to the obvious solution for the case when the elliptical supply contour becomes circular; when $\mu = 1$, the ellipse turns into a segment of the real axis of length 2c and the solution coincides with the analytical data for filtration from a rectilinear slit to the desired boundary of constant pressure difference [2].

The well-known analogy between steady processes of filtration and heat conduction allows a correspondence to be established between the modulus of the velocity and the heat flux, between the potential and the temperature, and between the liquid flow rate and the total heat flux. Then the results obtained may be interpreted as finding the optical form of the heat-source insulation in the form of an elliptical tube or a wire of elliptical cross sections ensuring the minimum heat losses at a fixed weight of the insulation. At the same time, with a fixed total heat flux and the given cross section of the insulation, its weight is a minimum.

The approach here outlined may also be used in solving other problems of filtration and heat conduction with unknown boundaries, for example, finding optimal insulation of two tubes or multiple-core wire.

2. Filtration According to a Power Law

Suppose that liquid filtration occurs according to a power law

$$\nabla h = -Cv^{s-1}\mathbf{v},$$

where C is a constant and $s > 0$. According to the theory of nonlinear filtration (see [5], for example), a second-order linear differential equation which the current function $\psi(v, \theta)$ satisfies may be written, introducing the locus velocity variables v and θ

$$v^2\frac{\partial^2\psi}{\partial v^2} + sv\frac{\partial\psi}{\partial v} + s\frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{6}$$

The solution of the following problem is now obtained. Suppose that a liquid filters from the heater borehole to the operational borehole, and the corresponding flow rates are q and −q. For convenience of the analysis, the boreholes are simulated by a point source and sink with the same flow rates as the boreholes. Where necessary, after solving this problem, the corresponding results for the formulation of the problem with two boreholes may be obtained by drawing circles of small radius with centers at the source and the sink and assuming that the pressure (difference) there is constant.

The flow pattern is assumed to be symmetric relative to the x and y axes. The flow region is bounded by the supply contour $h = 0$, and its form such that the filtrational flow rate q is a minimum for specified area of the region is found. According to the formulation of the problem, there is an additional condition $v = v_0$ at the supply contour Γ (Fig. 1b). The solution of Eq. (6) must be found in the region corresponding to the flow region in the θ, v plane. The method of conversion to the locus plane of the velocity is well known [5]. In the present case, this region is a halfstrip with a semiinfinite cut parallel to the v axis. The boundary conditions for the function $\psi(v, \theta)$ are as follows

$$\psi|_{AB} = \psi|_{A'B'} = 0, \quad \psi|_{A'DA} = q/2, \quad \left.\frac{\partial\psi}{\partial v}\right|_{BCB'} = 0. \tag{7}$$

Solving Eq. (6) with the conditions in Eq. (7), the current function is found, and then the formula of [5] is used to convert to the variables x, y and the position of the boundary Γ is determined.

The boundary problem is expediently solved by the straight-line method, preliminarily dividing the halfstrip with the cut into a rectangle and two halfstrips. The scheme for application of the method was described in [6], for example: the function ψ is written as a finite series, and determining the coefficients of this series reduces to solving a system of algebraic equations. Knowing the distance 2ℓ between the source and the sink, the dependence of the minimum flow rate on the area of the filtration region is found.

With linear filtration ($s = 1$), the problem may be precisely solved by means of the method of conformal mapping. Comparison with the precise solution allows the relative error of calculations by the straight-line method to be estimated. For the ten straight lines in the locus region, it is no greater than 3%. The results are shown in Fig. 3 and in Table 1. The minimum value of the area S/ℓ^2 for the two-dimensional problem is $\pi/2$, and the desired boundary takes the form of a circle of unit radius touching the ordinate. It is evident that, with increase in area of the filtration region, the dependence of the minimum flow rate on this area when $s = 1$ tends to linear form and the influence of the exponent s in the filtration law on the flow rate becomes increasingly significant.

NOTATION

G, filtration region; Γ, its unknown boundary; $\mathbf{v}$ and v, filtration rate and its modulus; h, pressure difference; H, pressure difference at Γ; q, flow rate; $\alpha = q/H$; n, external normal; x, y, Cartesian coordinates; D_u, auxiliary region; ξ, η, Cartesian coordinates in D_u, $u = \xi + i\eta$; ϕ, ψ, potential and current function; $W = \phi + i\psi$, complex potential; Ω, analytical function; ν, ε, its real and imaginary components; θ, angle between velocity vector and x axis; K, curvature; μ, eccentricity; c, major semiaxis; R, radius of curvature at the vertex of the ellipse; c_k, coefficients in series; s, power exponent in filtration law; 2ℓ, distance between boreholes.

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